An iterative approach to non-overdetermined inverse scattering at fixed energy

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Abstract. We propose an iterative approximate reconstruction algorithm for non-overdetermined inverse scattering at fixed energy E with incomplete data in dimension $d \geq 2$. In particular, we obtain rapidly converging approximate reconstructions for this inverse scattering for $E \to +\infty$.

1. Introduction

We consider the Schrödinger equation

$$H\psi = E\psi, \quad H = -\Delta + v(x), \quad x \in \mathbb{R}^d, \ d \geq 2, \ E > 0, \tag{1.1}$$

where

$$v \in L^\infty_\sigma(\mathbb{R}^d) \ \text{ for some } \ \sigma > d, \tag{1.2}$$

where

$$\begin{aligned} &L^\infty_\sigma(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \ \|u\|_\sigma < +\infty\}, \\ &\|u\|_\sigma = ess \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{\sigma/2} |u(x)|, \sigma \geq 0. \end{aligned} \tag{1.3}$$

For equation (1.1) we consider the classical scattering eigenfunctions ψ^+ specified by the following asymtotics as $|x| \to \infty$:

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k|\frac{x}{|x|}) + o\big(\frac{1}{|x|^{(d-1)/2}}\big), \\ &x \in \mathbb{R}^d, \ k \in \mathbb{R}^d, \ k^2 = E, \ c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}, \end{aligned} \tag{1.4}$$

where a priori unknown function $f = f(k,l)$, $k, l \in \mathbb{R}^d$, $k^2 = l^2 = E$, arising in (1.4) is the classical scattering amplitude for (1.1).

Given potential v, to determine ψ^+ and f one can use, in particular, the Lippmann-Schwinger integral equation

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G^+(x-y,k)v(y)\psi^+(y,k)dy, \\ &G^+(x,k) = -(2\pi)^{-d} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0}, \end{aligned} \tag{1.5}$$

and the formula

$$f(k,l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-ily} v(y)\psi^+(y,k)dy, \tag{1.6}$$

where $x, k, l \in \mathbb{R}^d$, $k^2 = l^2 = E > 0$; see, for example, [BS], [F2].

The scattering amplitude f at fixed energy $E > 0$ is defined on

$$\mathcal{M}_E = \{k \in \mathbb{R}^d,\ l \in \mathbb{R}^d :\ \ k^2 = l^2 = E\}, \quad E > 0. \tag{1.7}$$

Following [N8], in addition to f on $\mathcal{M}_E$ we consider also $f\big|_{\Gamma_E}$ and $f\big|_{\Gamma_E^\tau}$, where

$$\begin{aligned} &\Gamma_E = \{k = k_E(p),\ l = l_E(p) :\ p \in \mathcal{B}_{2\sqrt{E}}\},\\ &\Gamma_E^\tau = \{k = k_E(p),\ l = l_E(p) :\ p \in \mathcal{B}_{2\tau\sqrt{E}}\},\\ &k_E(p) = \frac{p}{2} + \eta_E(p), l_E(p) = -\frac{p}{2} + \eta_E(p), \end{aligned} \tag{1.8}$$

$$\mathcal{B}_r = \{p \in \mathbb{R}^d :\ |p| \le r\}, \quad r > 0, \tag{1.9}$$

where $E > 0$, $0 < \tau \le 1$, $d \ge 2$, and η_E is a piecewise continuous vector-function on $\mathcal{B}_{2\sqrt{E}}$ such that

$$\eta_E(p)p = 0, \quad \frac{p^2}{4} + (\eta_E(p))^2 = E, \quad p \in \mathcal{B}_{2\sqrt{E}}. \tag{1.10}$$

One can see that

$$\Gamma_E^\tau \subseteq \Gamma_E \subset \mathcal{M}_E,\ E > 0,\ 0 < \tau \le 1,\ d \ge 2. \tag{1.11}$$

In this work we continue studies on the following inverse scattering problems for equation (1.1) under assumptions (1.2):

Problem 1.1. Given scattering amplitude f on $\mathcal{M}_E$ at fixed $E > 0$, find potential v on $\mathbb{R}^d$ (at least approximately).

Problem 1.2. Given scattering amplitude f on Γ_E^τ at fixed E and τ, where $E > 0$, $0 < \tau \le 1$, find potential v on $\mathbb{R}^d$ (at least approximately).

In addition, one can see that

$$\begin{aligned} &dim\, \mathcal{M}_E = 2d - 2,\ dim\, \Gamma_E^\tau = dim\, \Gamma_E = dim\, \mathbb{R}^d = d \ \text{ for } \ d \ge 2,\\ &dim\, \mathcal{M}_E > d \ \text{ for } \ d \ge 3, \end{aligned} \tag{1.12}$$

where $E > 0$, $0 < \tau \le 1$. Therefore, Problem 1.1 is overdetermined for $d \ge 3$, whereas Problem 1.2 is non-overdetermined.

Problem 1.1 has a long history and there are many important results on this problem, see [ABR], [B], [BAR], [ChS], [E], [F1], [GHN], [G], [HH], [I], [IN], [N1]-[N5], [R], [S1], [VW], [W], [WY] and references therein. Note also that for spherical potentials v Problem 1.2 for $\tau = 1$ is reduced to Problem 1.1. However, to our knowledge, explicit considerations of Problem 1.2 were started only recently in [N8]. In addition, concerning known results for some other non-overdetermined multi-dimensional coefficient inverse problems, see [BK], [DKN], [ER1], [HN], [K], [M], [N6], [NS], [S2] and references therein.

Note also that Problems 1.1, 1.2 can be considered as examples of ill-posed problems; see [BK], [LRS] for an introduction to this theory.

In the present work we consider Problems 1.1, 1.2 assuming that

$$\begin{aligned} &v \text{ is a perturbation of some known background } v_0 \text{ satisfying (1.2)}, \\ &\text{where } v - v_0 \text{ is sufficiently regular on } \mathbb{R}^d \text{ and } \mathit{supp}\,(v - v_0) \subset D, \end{aligned} \tag{1.13}$$

where D is an open bounded domain (which is fixed a priori).

In particular, for Problem 1.2, under assumptions (1.13), we iteratively construct stable approximations $u_j(x, E)$ to the unknown $v(x)$, $x \in D$, where u_1 is a linear reconstruction in the Born approximation and u_j, $j \geq 2$, are non-linear approximate reconstructions from f on Γ_E^τ; see Subsections 3.2, 3.3. Our construction is based on direct scattering results (summarized in Section 2) and on standard Fourier analysis. In addition, our non-linear approximate reconstructions u_j are efficient in the sense that

$$\|u_j(\cdot, E) - v\|_{L^\infty(D)} = \varepsilon_j(E) \tag{1.14}$$

rapidly decay as $E \to +\infty$, for sufficiently regular $v - w$ and sufficiently large j. In particular,

$$\begin{aligned} &\varepsilon_j(E) = O(E^{-\alpha_j}), \quad \alpha_j = \left(1 - \left(\frac{n-d}{n}\right)^j\right)\frac{n-d}{2d}, \\ &\text{as } E \to +\infty, \quad j \geq 1, \end{aligned} \tag{1.15}$$

if $v - v_0$ is n-times smooth in $L^1(\mathbb{R}^d)$, $n > d$; see Theorem 3.1 of Subsection 3.4. Note that u_j and ε_j depend also on fixed τ of Problem 1.2.

In addition, in Subsection 3.5 we explain that the construction of u_j for each $j \in \mathbb{N}$ can be reduced to a finite number of explicit formulas; see Subsection 3.5 for details.

It is also important to note that f on $\Gamma_E^{\delta(E)}$ only is used in our iterative approximate reconstruction for Problem 1.2 at high energies E, where

$$\delta(E) = \tau E^{-(d-1)/(2d)}, \quad \tau \in]0, 1]. \tag{1.16}$$

In addition, $\delta(E) \to 0$ as $E \to +\infty$. Therefore, $\Gamma_E^{\delta(E)}$ is a very small part of $\Gamma_E^{\tau_1}$ for any fixed $\tau_1 \in]0, 1]$ for sufficiently large E. Therefore, our iterative approximate reconstruction can be viewed as a reconstruction result for Problem 1.2 with incomplete data.

Actually, the iterative approximate reconstruction of the present work complements related stability results of [N8].

In addition, the iterative reconstruction of the present work was also influenced by the iterative reconstruction of [N7] for quite different inverse scattering problem.

Let us consider also

$$\mathcal{M}_E^\tau = \{(k, l) \in \mathcal{M}_E : \ k - l \in \mathcal{B}_{2\tau\sqrt{E}}\}, \ E > 0, \ \tau \in]0, 1]. \tag{1.17}$$

Note that

$$\begin{aligned}
&\Gamma_E^\tau \subset \mathcal{M}_E^\tau \ \text{ for } \ \tau \in]0,1],\\
&\mathcal{M}_E^\tau \subset \mathcal{M}_E \ \text{ for } \ \tau \in]0,1[, \ \ \mathcal{M}_E^\tau = \mathcal{M}_E \ \text{ for } \ \tau = 1,\\
&dim\, \mathcal{M}_E^\tau = dim\, \mathcal{M}_E = 2d-2 \ \text{ for } \ \tau \in]0,1].
\end{aligned} \tag{1.18}$$

To our knowledge, no analog of the non-linear approximate reconstructions u_j, $j \geq 2$, was given in the iterature even for
Problem 1.1 with $\mathcal{M}_E$ replaced by $\mathcal{M}_E^\tau$ for some fixed $\tau \in]0,1[$, i.e. for the problem with much richer data than Problem 1.2 (and, especially, than Problem 1.2 with f given on $\Gamma_E^{\delta(E)}$ only, where $\delta(E)$ is defined by (1.16)).

On the other hand, for the case of Problem 1.1 with complete data, under assumptions (1.13), where $v - v_0$ is n-times smooth in $L^1(\mathbb{R}^d)$, $n > d$, and $v_0 \equiv 0$, the non-linear approximate reconstructions u_j, $j \geq 2$, of the present work are less precise than the approximate reconstructions of [N4], [N5] with the error term estimated as $O(E^{-s})$ in the uniform norm as $E \to +\infty$, where $s = (n-d)/2$ for $d = 2$, $s = (n-d-\delta)/2$ for any fixed arbitrary small $\delta > 0$ for $d = 3$. Indeed,

$$\alpha_j < \frac{n-d}{2d} < s, \tag{1.19}$$

where α_j are the numbers of (1.15), s is the number of [N4], [N5].

However, for the problem of approximate but stable finding v on $\mathbb{R}^d$ from f on $\Gamma_E^{\delta(E)}$ (or even on $\mathcal{M}_E^{\delta(E)}$) only, where $\delta(E)$ is defined by (1.16), our approximate reconstructions u_j for sufficiently large j are rather optimal with respect to their precision (1.14), (1.15) even in the framework of standards of the Born approximation.

Indeed, in the Born approximation (linear approximation near $v_0 \equiv 0$) the problem of finding v on $\mathbb{R}^d$ from f on $\mathcal{M}_E^\delta$, where $E > 0$, $\delta \in]0,1[$, $d \geq 2$, is reduced to finding v on $\mathbb{R}^d$ from its Fourier transform $\hat{v}$ on $\mathcal{B}_{2\delta\sqrt{E}}$, where

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \tag{1.20}$$

This linearized inverse scattering problem can be solved by the formula

$$\begin{aligned}
&v(x) = v_{appr}^{lin}(x,E,\delta) + v_{err}^{lin}(x,E,\delta), \quad x \in \mathbb{R}^d,\\
&v_{appr}^{lin}(x,E,\delta) = \int\limits_{\mathcal{B}_{2\delta\sqrt{E}}} e^{-ipx}\hat{v}(p)dp,\\
&v_{err}^{lin}(x,E,\delta) = \int\limits_{\mathbb{R}^d \backslash \mathcal{B}_{2\delta\sqrt{E}}} e^{-ipx}\hat{v}(p)dp.
\end{aligned} \tag{1.21}$$

In addition, we have that:

$$\begin{aligned}
&\varepsilon(E,\delta) \stackrel{\text{def}}{=} \|v_{err}^{lin}(x,E,\delta)\|_{L^\infty(\mathbb{R}^d)} =\\
&O((\delta\sqrt{E})^{-(n-d)}) \ \text{ if } \ \delta\sqrt{E} \to +\infty
\end{aligned} \tag{1.22}$$

under the assuption that v is n-times smooth in $L^1(\mathbb{R}^d)$, $n > d$. In addition,

$$\varepsilon(E, \delta(E)) = O((\delta(E)\sqrt{E})^{-(n-d)}) = O(E^{-\alpha}), \quad \alpha = \frac{n-d}{2d}, \ E \to +\infty \tag{1.23}$$

for $\delta(E)$ given by (1.16).

Finally, one can see that $\alpha_j \to \alpha$ for $j \to +\infty$, where α_j are the numbers of (1.15) and α is the number of the Born approximation estimates (1.22), (1.23).

2. Preliminaries of direct scattering

In this section we summarize some results related with the Lippmann- Schwinger integral equation (1.5) for the scattering eigenfunctions ψ^+ and with formula (1.6) for the scattering amplitude f.

It is convenient to write the Lippmann-Schwinger integral equation (1.5) as

$$(I - A(k))\varphi(\cdot, k) = e(\cdot, k), \tag{2.1}$$

where

$$\begin{aligned} &\varphi(x, k) = \Lambda^{-\sigma/2}\psi^+(x, k), \quad e(x, k) = \Lambda^{-\sigma/2}e^{ikx}, \\ &A(k) = \Lambda^{-\sigma/2}G^+(k)\Lambda^{-\sigma/2}(\Lambda^\sigma v), \ k \in \mathbb{R}^d\backslash\{0\}, \ x \in \mathbb{R}^d, \end{aligned} \tag{2.2}$$

where I is the identity operator, Λ denotes the multiplication operator by the functions $(1 + |x|^2)^{1/2}$, G^+ denotes the integral operator with the Schwartz kernel $G^+(x - y, k)$ of (1.5), v is the multiplication operator by the function $v(x)$, σ is the number of (1.2). In addition, we recall that the following estimate holds:

$$\begin{aligned} &\|\Lambda^{-s}G^+(k)\Lambda^{-s}\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \le a_0(d, s)|k|^{-1}, \\ &k \in \mathbb{R}^d, \ |k| \ge 1, \quad \text{for} \quad s > 1/2, \end{aligned} \tag{2.3}$$

see [E], [J] and references therein.

Using (2.3) one can see that

$$\|A(k)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \le a_0(d, \sigma/2)\|v\|_\sigma |k|^{-1}, \quad k \in \mathbb{R}^d, \ |k| \ge 1, \tag{2.4}$$

where $\| \cdot \|_\sigma$ is defined in (1.3).

As a corollary of (2.1), (2.2), (2.4), we have that

$$\begin{aligned} &\|\varphi(\cdot, k) - \sum_{j=0}^{m}(A(k))^j e(\cdot, k)\|_{L^2(\mathbb{R}^d)} \le 2\left(\frac{a_0(d, \sigma/2)\|v\|_\sigma}{|k|}\right)^{m+1} c_1(d) \\ &\text{for} \ \ k \in \mathbb{R}^d, \ |k| \ge \rho_1(d, \sigma, \|v\|_\sigma), \quad m \in \mathbb{N} \cup 0, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &c_1(d, \sigma) = \|e(\cdot, k)\|_{L^2(\mathbb{R}^d)} = \left(\int\limits_{\mathbb{R}^d} \frac{dx}{(1 + |x|^2)^{\sigma/2}}\right)^{1/2}, \\ &\rho_1(d, \sigma, N) = \max(2a_0(d, \sigma/2)N, 1). \end{aligned} \tag{2.6}$$

Actually, formula (2.5) is a well-known method for solving the Lippmann-Schwinger integral equation (1.5) for sufficiently high energies.

Let now φ_i, A_i denote φ, A for $v = v_i$ satisfying (1.2), where $i = 1, 2$. Using the identity

$$\begin{aligned}&(I - A_2(k))^{-1} - (I - A_1(k))^{-1} = \\ &(I - A_2(k))^{-1}(A_2(k) - A_1(k))(I - A_1(k))^{-1},\end{aligned} \tag{2.7}$$

one can see that

$$\begin{aligned}&\varphi_2(\cdot, k) - \varphi_1(\cdot, k) = \\ &(I - A_2(k))^{-1}(A_2(k) - A_1(k))(I - A_1(k))^{-1}e(\cdot, k).\end{aligned} \tag{2.8}$$

Using (2.2), (2.3), (2.4), (2.8), we obtain that

$$\begin{aligned}&\|\varphi_2(\cdot, k) - \varphi_1(\cdot, k)\|_{L^2(\mathbb{R}^d)} \leq 4a_0(d, \sigma/2)\|v_2 - v_1\|_\sigma c_1(d, \sigma)|k|^{-1} \\ &\text{for} \ \ k \in \mathbb{R}^d, \ |k| \geq \rho_1(d, \sigma, N),\end{aligned} \tag{2.9}$$

where c_1, ρ_1 are defined in (2.6), $\|v_i\|_\sigma \leq N$ for $i = 1, 2$.

Due to (1.6), we have also that

$$\hat{v}(k - l) = f(k, l) - (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-ilx}v(x)(\psi^+(x, k) - e^{ikx})dx, \ \ (k, l) \in \mathcal{M}_E, \tag{2.10}$$

where $\hat{v}$ is defined by (1.20).

In particular, as a corollary of (2.10) and (2.5) for $m = 0$, we have that

$$\begin{aligned}&|f(k, l) - \hat{v}(k - l)| \leq 2(2\pi)^{-d}a_0(d, \sigma/2)(c_1(d, \sigma)\|v\|_\sigma)^2 E^{-1/2}, \\ &(k, l) \in \mathcal{M}_E, \ \ E^{1/2} \geq \rho_1(d, \sigma, \|v\|_\sigma),\end{aligned} \tag{2.11}$$

where c_1, ρ_1 are defined in (2.6).

Our iterative reconstruction algorithm for Problem 1.2 is based on formulas (2.5), (2.9), (2.10), (2.11) and is presented in the next section.

3. Iterative approximate reconstruction for Problem 1.2

3.1. Assumptions and notations. We consider

$$\begin{aligned}&W^{n,1}(\mathbb{R}^d) = \{u : \ \partial^J u \in L^1(\mathbb{R}^d), \ |J| \leq n\}, \\ &\|u\|_{n,1} = \max_{|J| \leq n} \|\partial^J u\|_{L^1(\mathbb{R}^d)}, \ n \in \mathbb{N} \cup 0,\end{aligned} \tag{3.1}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \ |J| = \sum_{i=1}^d J_i, \ \partial^J u(x) = \frac{\partial^{|J|}u(x)}{\partial x_1^{J_1} \ldots \partial x_d^{J_d}}.$$

We assume that v satisfies (1.13), where the assumption that $v - v_0$ is sufficiently regular is specified as

$$v - v_0 \in W^{n,1}(\mathbb{R}^d) \ \text{ for some } \ n > d. \tag{3.2}$$

We set

$$w = v - v_0 \tag{3.3}$$

and consider the decompositions

$$\begin{aligned} v(x) &= v^+(x,\kappa) + v^-(x,\kappa), \\ v_0(x) &= v_0^+(x,\kappa) + v_0^-(x,\kappa), \\ w(x) &= w^+(x,\kappa) + w^-(x,\kappa), \end{aligned} \tag{3.4}$$

where $x \in D$, $\kappa > 0$,

$$\begin{aligned} u^+(x,\kappa) &= \int\limits_{p\in\mathbb{R}^d,\ |p|\le\kappa} e^{-ipx}\hat u(p)\,dp, \\ u^-(x,\kappa) &= \int\limits_{p\in\mathbb{R}^d,\ |p|>\kappa} e^{-ipx}\hat u(p)\,dp, \\ \hat u(p) &= (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{ipx}u(x)\,dx \end{aligned} \tag{3.5}$$

for $u = v, v_0, w$.

Due to (3.3)-(3.5), we have that

$$v(x) = v^+(x,\kappa) + v_0^-(x,\kappa) + w^-(x,\kappa), \tag{3.6}$$
$$v^+(x,\kappa) = v_0^+(x,\kappa) + w^+(x,\kappa), \tag{3.7}$$

where $x \in D$, $\kappa > 0$.

3.2. Reconstruction in the Born approximation. We define

$$v_1(x,E,\tau_1) \stackrel{\text{def}}{=} v_1^+(x,E,\tau_1) + v_0^-(x, 2\tau_1\sqrt{E}), \tag{3.8a}$$
$$v_1^+(x,E,\tau_1) \stackrel{\text{def}}{=} \int\limits_{p\in\mathbb{R}^d,\ |p|\le 2\tau_1\sqrt{E}} e^{-ipx} f(k_E(p), l_E(p))dp, \tag{3.8b}$$
$$x \in D, \quad 0 < \tau_1 \le \tau, \quad E > 0,$$

where v_0^- is the function of (3.4), (3.6), f is the scattering amplitude for v, and k_E, l_E are defined in (1.8). Actually, v_1 of (3.8a) is a reconstruction in the Born approximation for Problem 1.2.

Lemma 3.1. *Let v satisfy (1.13), (3.2) and $\|v\|_\sigma \le M_1$, $\|v - v_0\|_{n,1} \le M_2$. Then:*

$$|v_1(x,E,\tau_1) - v(x)| \le c_2(d,\sigma)M_1^2\frac{(2\tau_1\sqrt{E})^d}{\sqrt{E}} + \frac{c_3(d,n)M_2}{(2\tau_1\sqrt{E})^{n-d}} \tag{3.9}$$

for $x \in D$, $0 < \tau_1 \le \tau$, $\sqrt{E} \ge \rho_1(d,\sigma,M_1)$,

$$|v_1(x,E,\tau_1(E))-v(x)| \le \left(c_2(d,\sigma)M_1^2(2\tau)^d + \frac{c_3(d,n)M_2}{(2\tau)^{n-d}}\right)E^{-(n-d)/(2n)}$$
$$\text{for } \ x\in D,\quad \tau_1(E)=\tau\, E^{-(n-1)/(2n)},\ \sqrt{E}\ge\rho_1(d,\sigma,M_1), \tag{3.10}$$

where $0<\tau\le 1$, v_1 is defined by (3.8), ρ_1 is defined in (2.6), and $c_2=c_2(d,\sigma)$, $c_3=c_3(d,n)$ are some positive constants.

Lemma 3.1 is proved in Section 4.

Under the assumptions of Lemma 3.1,

$$\begin{aligned} &u_1(x,E)=v_1(x,E,\tau_1(E)),\quad x\in D,\\ &u_1(x,E)=v_0(x),\quad x\in\mathbb{R}^d\backslash D, \end{aligned} \tag{3.11}$$

can be considered as an optimal reconstruction in the Born approximation for Problem 1.2 with respect to the error decay in $L^\infty(D)$ as $E\to+\infty$.

3.3. Iterative step. The iterative step of our reconstruction is based, in particular, on the following lemma:

Lemma 3.2. *Let v satisfy (1.13), f be the scattering amplitude of v, and $v_{appr}(\cdot,E)$ be an approximation to v such that*

$$|v_{appr}(x,E)-v(x)|\le\beta E^{-\alpha},\ x\in D,\ \sqrt{E}\ge\rho_1(d,\sigma,N), \tag{3.12a}$$

$$v_{appr}(x,E)\equiv v_0(x),\quad x\in\mathbb{R}^d\backslash D, \tag{3.12b}$$

for some α, $\beta>0$ and some N such that

$$\|v\|_\sigma\le N,\quad \|v_{appr}(\cdot,E)\|_\sigma\le N,\quad \sqrt{E}\ge\rho_1(d,\sigma,N), \tag{3.13}$$

where ρ_1 is defined in (2.6). Then the following estimate holds:

$$\begin{aligned} &|f(k,l)-f_{appr}(k,l)+\hat v_{appr}(k-l,E)-\hat v(k-l)|\le\\ &(2\pi)^{-d}a_0(d,\sigma/2)c_1(d,\sigma)c_4(D,\sigma)N\beta E^{-\alpha-(1/2)},\\ &(k,l)\in\mathcal{M}_E,\ E^{1/2}\ge\rho_1(d,\sigma,N), \end{aligned} \tag{3.14}$$

where f_{appr} is the scattering amplitude for v_{appr}, $\hat v_{appr}(\cdot,E)$ is the Fourier transform of $v_{appr}(\cdot,E)$, $\hat v$ is the Fourier transform of v, (see definition (1.20)), $c_4(D,\sigma)$ is given by (4.12).

Lemma 3.2 is proved in Section 4.

In the iterative step of our reconstruction we assume that v satisfies (1.13), (3.2) and $\|v\|_\sigma\le M_1$, $\|v-v_0\|_{n,1}\le M_2$, as in lemma 3.1.

Note that $v_{appr}=u_1$ of (3.11) satisfies (3.12), (3.13) for $\alpha=\alpha_1$, $\beta=\beta_1$, $N=N_1$, where

$$\begin{aligned} &\alpha_1=\frac{n-d}{2n},\quad \beta_1=c_2(d,\sigma)M_1^2(2\tau)^d+\frac{c_3(d,n)M_2}{(2\tau)^{n-d}},\\ &N_1=M_1+c_5(D,\sigma)\beta_1(\rho_1(d,\sigma,M_1))^{-\alpha_1},\quad c_5(D,\sigma)=\sup_{x\in D}(1+|x|^2)^{\sigma/2}. \end{aligned} \tag{3.15}$$

Then proceeding from the approximation $v_{appr} = u_j(\cdot, E)$ with number j, satisfying (3.12), (3.13) for $\alpha = \alpha_j$, $\beta = \beta_j$, $N = N_j$, the approximation $v_{appr} = u_{j+1}(\cdot, E)$ with number $j+1$ is constructed as follows:

(1) We find the scattering amplitude f_j and the Fourier transform $\hat{u}_j(\cdot, E)$ for $u_j(\cdot, E)$, where $E^{1/2} \geq \rho_1(d, \sigma, N_j)$;

(2) In a similar way with (3.8), we define

$$v_{j+1}(x, E, \tau_{j+1}) \stackrel{\text{def}}{=} v^+_{j+1}(x, E, \tau_{j+1}) + v_0^-(x, 2\tau_{j+1}\sqrt{E}), \qquad (3.16a)$$

$$\begin{aligned} &v^+_{j+1}(x, E, \tau_{j+1}) \stackrel{\text{def}}{=} \int\limits_{p\in\mathbb{R}^d,\ |p|\leq 2\tau_{j+1}\sqrt{E}} e^{-ipx} \times \\ &(f(k_E(p), l_E(p)) - f_j(k_E(p), l_E(p)) + \hat{u}_j(k_E(p) - l_E(p), E))dp, \qquad (3.16b) \\ &x \in D, \quad 0 < \tau_{j+1} \leq \tau, \quad E^{1/2} \geq \rho_1(d, \sigma, N_j), \end{aligned}$$

where v_0^-, f, k_E, l_E are the same that in (3.8);

(3) Finally, in a similar way with (3.11), we define

$$\begin{aligned} &u_{j+1}(x, E) = v_{j+1}(x, E, \tau_{j+1}(E)),\ x \in D, \\ &u_{j+1}(x, E) = v_0(x),\ x \in \mathbb{R}^d \backslash D, \qquad (3.17) \\ &\tau_{j+1}(E) = \tau\ E^{-(n-1-2\alpha_j)/(2n)}, \end{aligned}$$

where $E^{1/2} \geq \rho_1(d, \sigma, N_j)$.

In addition, we have the following lemma:

Lemma 3.3. *Under the assumptions of our iterative step, the following estimates hold:*

$$\begin{aligned} &|v_{j+1}(x, E, \tau_{j+1}) - v(x)| \leq c_6(D, \sigma)N_j\beta_j \frac{(2\tau_{j+1}E^{1/2})^d}{E^{\alpha_j+(1/2)}} + \frac{c_3(d, n)M_2}{(2\tau_{j+1}E^{1/2})^{n-d}} \qquad (3.18) \\ &\text{for } \ x \in D, \quad 0 < \tau_{j+1} \leq \tau \leq 1, \quad E^{1/2} \geq \rho_1(d, \sigma, N_j), \quad j \in \mathbb{N}, \end{aligned}$$

$$\begin{aligned} &|u_{j+1}(x, E) - v(x)| \leq \left(c_6(D, \sigma)N_j\beta_j(2\tau)^d + \frac{c_3(d, n)M_2}{(2\tau)^{n-d}}\right)E^{-(1+2\alpha_j)(n-d)/(2n)} \qquad (3.19) \\ &\text{for } \ x \in D, \quad E^{1/2} \geq \rho_1(d, \sigma, N_j), \quad j \in \mathbb{N}, \end{aligned}$$

where v_{j+1}, u_{j+1} are defined by (3.16), (3.17), ρ_1 is defined in (2.6).

Lemma 3.3 is proved in Section 5.

In addition, $u_{j+1}(\cdot, E)$ of (3.17) satisfies (3.12), (3.13) for $\alpha = \alpha_{j+1}$, $\beta = \beta_{j+1}$, $N = N_{j+1}$, where

$$\begin{aligned} &\alpha_{j+1} = \frac{(1 + 2\alpha_j)(n - d)}{2n}, \quad \beta_{j+1} = c_6(D, \sigma)N_j\beta_j(2\tau)^d + \frac{c_3(d, n)M_2}{(2\tau)^{n-d}}, \qquad (3.20) \\ &N_{j+1} = M_1 + c_5(D, \sigma) \max_{1\leq i\leq j} \beta_i(\rho_1(d, \sigma, N_i))^{-\alpha_i}, \quad j \in \mathbb{N}. \end{aligned}$$

Note also that

$$N_{j_1} \le N_{j_2}, \quad \rho_1(d,\sigma,N_{j_1}) \le \rho_1(d,\sigma,N_{j_2}) \ \text{ for } \ 1 \le j_1 \le j_2. \tag{3.21}$$

3.4. Final theorem. Proceeding from lemmas 3.1, 3.2, 3.3 and formulas (3.15), (3.20), (3.21) we obtain the following theorem:

Theorem 3.1. *Let v satisfy (1.13), (3.1) and $\|v\|_\sigma \le M_1$, $\|v - v_0\|_{n,1} \le M_2$. Let $u_j(\cdot, E)$ be constructed from $f\big|_{\Gamma_E^{\delta(E)}}$ by the iterations of subsections 3.2, 3.3, where f is the scattering amplitude for v, $\delta(E) = \tau E^{-(d-1)/(2d)}$, $0 < \tau \le 1$, and $j \in \mathbb{N}$; see formulas (3.11), (3.16), (3.17). Then the following estimates hold:*

$$\|u_j(\cdot,E) - v\|_{L^\infty(D)} \le \beta_j E^{-\alpha_j} \ \text{ for } \ E^{1/2} \ge \rho_1(d,\sigma,N_{j-1}), \tag{3.22}$$

where

$$\alpha_j = \left(1 - \left(\frac{n-d}{n}\right)^j\right)\frac{n-d}{2d}, \tag{3.23}$$

$\beta_j = \beta_j(M_1, M_2, D, \sigma, n, \tau)$, $N_j = N_j(M_1, M_2, D, \sigma, n, \tau)$ are constructed recurrently via (3.15), (3.20) (with $N_0 = M_1$).

Theorem 3.1 is proved in Section 5.

3.5. Explicit formulas. One can see that:

(1) $u_1(\cdot, E)$ is constructed by explicit formulas from $f\Big|_{\Gamma_E^{\delta(E)}}$ and v_0 for $\sqrt{E} \ge \rho_1(d,\sigma,M_1)$ (see Subsection 3.2 and formula (5.9)) and

(2) $u_{j+1}(\cdot, E)$ is constructed by explicit formulas from $u_j(\cdot,E)$, $f\Big|_{\Gamma_E^{\delta(E)}}$, $f_j\Big|_{\Gamma_E^{\delta(E)}}$ and v_0 for $\sqrt{E} \ge \rho_1(d,\sigma,N_j)$ and each $j \in \mathbb{N}$ (see Subsection 3.3 and formula (5.9)). However, f_j is constructed from u_j via (1.5), (1.6) with $\psi^+ = \psi_j^+$, $v = v_j$, where (1.5) is not yet an explicit formula for ψ_j^+.

Thus the construction of $u_j(\cdot,E)$, $j \ge 2$, of Subsections 3.3, 3.4 is not reduced yet to explicit formulas. In order to have a similar construction involving explicit formulas only we can proceed as follows. Instead of u_j, $j \ge 2$, we can construct $\tilde u_j(\cdot,E)$, $j \ge 2$, $\sqrt{E} \ge \rho_1(d,\sigma,M_1)$, via (3.16), (3.17) with f_j, $\hat u_j$, u_{j+1} replaced by $\tilde f_j^{appr}$, $\hat{\tilde u}_j$, $\tilde u_{j+1}$, where $\hat{\tilde u}_j$ is the Fourier transform of $\tilde u_j$ as before, whereas $\tilde f_j^{appr}$ is the approximation to the scattering amplitude $\tilde f_j$ of $\tilde u_j$, defined as follows. Proceeding from (1.6), (1.5), (2.2), (2.5), we define

$$\tilde f_j^{appr}(k,l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-ily}\Lambda^{\sigma/2} v(y)\tilde\varphi_j^{appr}(y,k)dy, \tag{3.24}$$

$$\tilde\varphi_j^{appr}(\cdot,k) = \sum_{\nu=0}^{m_j} (\tilde A_j(k))^\nu e(\cdot,k), \tag{3.25}$$

where $(k,l) \in \mathcal{M}_E$, m_j is the minimal natural number such that $m_j \geq 2\alpha_j$, $\tilde{A}_j(k)$ is defined as $A(k)$ of (2.2) with v replaced by $\tilde{u}_j$, $e(\cdot,k)$ is defined in (2.2). Here α_j is the number of (3.23), $j \in \mathbb{N}$.

Using estimate (2.5) with v replaced by $\tilde{u}_j$ and using the proof of Lemma 3.3, one can show that

$$\|\tilde{u}_j(\cdot,E) - v\|_{L^\infty(D)} = O(E^{-\alpha_j}) \text{ as } E \to +\infty, \tag{3.26}$$

as for the initial u_j. However, now because of the finite sum in (3.25), $\tilde{u}_j$ is constructed via a finite number of explicite formulas for each $j \geq 2$.

4. Proofs of Lemmas 3.1 and 3.2

Proof of lemma 3.1. Due to (3.6), we have that

$$v(x) = v^+(x, 2\tau_1\sqrt{E}) + v_0^-(x, 2\tau_1\sqrt{E}) + w^-(x, 2\tau_1\sqrt{E}), \quad x \in D. \tag{4.1}$$

Due to (3.8), (4.1), (3.5), we have that

$$\begin{gathered} v_1(x,E,\tau_1) - v(x) = \delta_1^+ v(x,E,\tau_1) + \delta_1^- v(x,E,\tau_1), \\ \delta_1^+ v(x,E,\tau_1) \stackrel{\text{def}}{=} \int\limits_{p\in\mathbb{R}^d,\ |p|\leq 2\tau_1\sqrt{E}} e^{-ipx}(f(k_E(p), l_E(p)) - \hat{v}(p))dp, \\ \delta_1^- v(x,E,\tau_1) \stackrel{\text{def}}{=} -w^-(x, 2\tau_1\sqrt{E}), \quad x \in D. \end{gathered} \tag{4.2}$$

Using (2.11) and the definitions of $k_E(p)$, $l_E(p)$ of (1.8) we obtain that

$$\begin{gathered} |\delta_1^+(x,E,\tau_1)| \leq 2(2\pi)^{-d} a_0(d,\sigma/2)(c_1(d,\sigma)\|v\|_\sigma)E^{-1/2}\times \\ (2\tau_1\sqrt{E})^d|\mathcal{B}_1|, \quad x \in D, \quad \sqrt{E} \geq \rho_1(d,\sigma,\|v\|_\sigma), \end{gathered} \tag{4.3}$$

where $|\mathcal{B}_1|$ denotes the standard Euclidean volume of $\mathcal{B}_1$, i.e.

$$|\mathcal{B}_1| = \int\limits_{p\in\mathbb{R}^d,\ |p|\leq 1} dp. \tag{4.4}$$

In order to estimate $\delta_1^- v(x,E,\tau_1)$ we use that if $w \in W^{n,1}(\mathbb{R}^d)$, then

$$|\hat{w}(p)| \leq a_1(n,d)\|w\|_{n,1}(1+|p|)^{-n}, \quad p \in \mathbb{R}^d. \tag{4.5}$$

Using the definition of w^- of (3.3)-(3.5) and estimate (4.5) we obtain that

$$\begin{gathered} |\delta_1^-(x,E,\tau_1)| \leq \int\limits_{p\in\mathbb{R}^d,\ |p|\geq 2\tau_1\sqrt{E}} \frac{a_1(n,d)\|v-v_0\|_{n,1}}{(1+|p|)^n} dp \leq \\ |\mathbb{S}^{d-1}| \frac{a_1(n,d)\|v-v_0\|_{n,1}}{n-d} \frac{1}{(2\tau_1\sqrt{E})^{n-d}}, \quad x \in D, \end{gathered} \tag{4.6}$$

where $|\mathbb{S}^{d-1}|$ denotes the standard Euclidean volume of $\mathbb{S}^{d-1}$, i.e.

$$|\mathbb{S}^{d-1}| = \int\limits_{\theta\in\mathbb{S}^{d-1}} d\theta. \tag{4.7}$$

Estimate (3.9) follows from (4.2), (4.3), (4.6). Estimate (3.10) follows from (3.9).

Lemma 3.1 is proved.

Proof of Lemma 3.2. Using formula (1.6) for the scattering amplitude we obtain that

$$\begin{aligned}
&f(k,l) = \hat{v}(k-l) + \delta f(k,l),\\
&\delta f(k,l) \stackrel{\text{def}}{=} (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-ilx}v(x)(\psi^+(x,k)-e^{ikx})dx,\\
&f_{appr}(k,l) = \hat{v}(k-l,E) + \delta f_{appr}(k,l),\\
&\delta f_{appr}(k,l) \stackrel{\text{def}}{=} (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-ilx}v_{appr}(x,E)(\psi^+_{appr}(x,k)-e^{ikx})dx,\\
&(k,l)\in\mathcal{M}_E,\quad \sqrt{E}\ge\rho_1(d,\sigma,N),
\end{aligned} \tag{4.8}$$

where ψ^+_{appr} denotes the scattering solutions for v_{appr}.

In addition, we have that

$$\begin{aligned}
&\delta f(k,l) - \delta f_{appr}(k,l) =\\
&(2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-ilx}(v(x)-v_{appr}(x,E))(\psi^+(x,k)-e^{ikx})dx+\\
&(2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-ilx}v_{appr}(x,E)(\psi^+(x,k)-\psi^+_{appr}(x,k))dx,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&|\delta f(k,l) - \delta f_{appr}(k,l)| \le\\
&(2\pi)^{-d}\int\limits_{\mathbb{R}^d} \Lambda^{\sigma/2}|v(x)-v_{appr}(x,E)|\Lambda^{-\sigma/2}|\psi^+(x,k)-e^{ikx}|dx+\\
&(2\pi)^{-d}\int\limits_{\mathbb{R}^d} \Lambda^{\sigma/2}|v_{appr}(x,E)|\Lambda^{-\sigma/2}|\psi^+(x,k)-\psi^+_{appr}(x,k)|dx,
\end{aligned} \tag{4.10}$$

where Λ denotes the function $(1+|x|^2)^{1/2}$. Using (4.10), (2.5) for $m=0$, (2.6), (2.9) for $v_1 = v$, $v_2 = v_{appr}$, (3.12), (3.13), we obtain that

$$\begin{aligned}
&|\delta f(k,l) - \delta f_{appr}(k,l)| \le\\
&2(2\pi)^{-d}a_0(d,\sigma/2)c_1(d,\sigma)\|\Lambda^{\sigma/2}\|_{L^2(D)}N\beta\, E^{-\alpha-(1/2)}+\\
&4(2\pi)^{-d}a_0(d,\sigma/2)c_1(d,\sigma)\|\Lambda^{-\sigma/2}\|_{L^2(\mathbb{R}^d)}\|\Lambda^{\sigma}\|_{L^\infty(D)}N\beta\, E^{-\alpha-(1/2)} =\\
&(2\pi)^{-d}a_0(d,\sigma/2)c_1(d,\sigma)c_4(D,\sigma)N\beta\, E^{-\alpha-(1/2)}
\end{aligned} \tag{4.11}$$

for $(k,l) \in \mathcal{M}_E$, $\sqrt{E} \ge \rho_1(d,\sigma,N)$, where

$$c_4(D,\sigma) = 2\|\Lambda^{\sigma/2}\|_{L^2(D)} + 4\|\Lambda^{-\sigma/2}\|_{L^2(\mathbb{R}^d)}\|\Lambda^{\sigma}\|_{L^\infty(D)}. \tag{4.12}$$

Note also that

$$\|\Lambda^{-\sigma/2}\|_{L^2(\mathbb{R}^d)} = c_1(d,\sigma), \quad \|\Lambda^{\sigma/2}\|_{L^2(D)} \le \|\Lambda^{\sigma}\|_{L^\infty(D)}\|\Lambda^{-\sigma/2}\|_{L^2(\mathbb{R}^d)}.$$

Estimate (3.14) follows from (4.8), (4.11).
Lemma 3.2 is proved.

5. Proofs of Lemma 3.3 and Theorem 3.1

Proof of Lemma 3.3. In a completely similar way with (4.1) we have that

$$v(x) = v^+(x, 2\tau_{j+1}\sqrt{E}) + v_0^-(x, 2\tau_{j+1}\sqrt{E}) + w^-(x, 2\tau_{j+1}\sqrt{E}), \quad x \in D, \tag{5.1}$$

where v^+, v_0^-, w^- are the functions of (3.4), (3.6). Due to (3.16), (5.1), (3.5), in a similar way with (4.2) we have that

$$\begin{aligned}
&v_{j+1}(x,E,\tau_{j+1}) - v(x) = \delta^+_{j+1}v(x,E,\tau_{j+1}) + \delta^-_{j+1}v(x,E,\tau_{j+1}),\\
&\delta^+_{j+1}v(x,E,\tau_{j+1}) \stackrel{\mathrm{def}}{=} \int\limits_{p\in\mathbb{R}^d,\ |p|\le 2\tau_{j+1}\sqrt{E}} e^{-ipx}\times\\
&(f(k_E(p),l_E(p)) - f_j(k_E(p),l_E(p)) + \hat{u}_j(k_E(p)-l_E(p),E) - \hat{v}(p))dp,\\
&\delta^-_{j+1}v(x,E,\tau_{j+1}) \stackrel{\mathrm{def}}{=} -w^-(x, 2\tau_{j+1}\sqrt{E}), \quad x \in D.
\end{aligned} \tag{5.2}$$

Using (3.14) for $f_{appr} = f_j$, $\hat{v}_{appr} = \hat{u}_j$, $\alpha = \alpha_j$, $\beta = \beta_j$, $N = N_j$ and using the definitions of $k_E(p)$, $l_E(p)$ of (1.8), in a similar way with (4.3) we have that

$$\begin{aligned}
&|\delta^+_{j+1}v(x,E,\tau_{j+1})| \le c_6(D,\sigma)N_j\beta_j E^{-\alpha_j-(1/2)}(2\tau_{j+1}\sqrt{E})^d,\\
&x \in D, \quad \sqrt{E} \ge \rho_1(d,\sigma,N_j),
\end{aligned} \tag{5.3}$$

where

$$c_6(D,\sigma) = (2\pi)^{-d}a_0(d,\sigma/2)c_1(d,\sigma)c_4(D,\sigma)|\mathcal{B}_1|. \tag{5.4}$$

In addition, in a completely similar way with (4.6) we have that

$$|\delta^-_{j+1}v(x,E,\tau_{j+1})| \le |\mathbb{S}^{d-1}|\frac{a_1(n,d)\|v-v_0\|_{n,1}}{n-d}\frac{1}{(2\tau_{j+1}\sqrt{E})^{n-d}}, \quad x \in D. \tag{5.5}$$

Estimate (3.18) follows from (5.3), (5.5). Estimate (3.19) follows from (3.18), (3.17) and from the identities

$$\begin{aligned}
&E^{-(n-1-2\alpha_j)/(2n)}E^{1/2} = E^{(1+2\alpha_j)/(2n)},\\
&E^{d(1+2\alpha_j)/(2n)}E^{-\alpha_j-(1/2)} = E^{-(1+2\alpha_j)(n-d)/(2n)}.
\end{aligned} \tag{5.6}$$

This completes the proof of Lemma 3.3.

Proof of Theorem 3.1. Due to (3.15), (3.20), we have that

$$\alpha_1 = \frac{n-d}{2n}, \quad \alpha_{j+1} = \frac{n-d}{2n} + \alpha_j \frac{n-d}{n}, \quad j \in \mathbb{N}. \tag{5.7}$$

Formulas (5.7) imply (3.23). Indeed, the sequence α_j, $j \in \mathbb{N}$, is uniquely defined by (5.7) and α_j of (3.23) satisfy (5.7). In particular,

$$\begin{aligned} &\frac{n-d}{2n} + \left(1 - \left(\frac{n-d}{n}\right)^j\right)\frac{n-d}{2d}\frac{n-d}{n} = \\ &\frac{n-d}{2n} + \frac{n-d}{2d}\frac{n-d}{n} - \left(\frac{n-d}{n}\right)^{j+1}\frac{n-d}{2d} = \\ &\left(1 - \left(\frac{n-d}{n}\right)^{j+1}\right)\frac{n-d}{2d}, \quad j \in \mathbb{N}. \end{aligned} \tag{5.8}$$

Next, due to (5.7), (3.23) and due to the definition of $\tau_j(E)$, $j \in \mathbb{N}$, of (3.10), (3.17) we have that

$$\tau_{j_1}(E) = \tau E^{-(n-1-2\alpha_{j_1})/(2n)} \leq \tau_{j_2}(E) = \tau E^{-(n-1-2\alpha_{j_2})/(2n)} \leq \delta(E) = \tau E^{-(d-1)/(2d)} \tag{5.9}$$

for $0 < \tau \leq 1$, $1 \leq j_1 \leq j_2$, $E \geq 1$.

Using the definition of u_j, $j \in \mathbb{N}$, of (3.11), (3.17) and using inequalities of (3.21) and (5.9) we obtain that $u_j(\cdot, E)$ are correctly defined in terms of $f\Big|_{\Gamma_E^{\delta(E)}}$ for $E^{1/2} \geq \rho_1(d, \sigma, N_{j-1})$, $j \in \mathbb{N}$.

Finally, estimates (3.22) follow from estimates (3.10), (3.19) and formulas (3.15), (3.20).

This completes the proof of Theorem 3.1.

Acknowledgements

This work was partially supported by TFP No 14.A18.21.0866 of Ministry of Education and Sciences of Russian Federation (at Faculty of Control and Applied Mathematics of Moscow Institute of Physics and Technology).

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